Exercise 28

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x \to -6^+} \sqrt[8]{6+x} = 0$$

Solution

According to Definition 4, proving this limit is logically equivalent to proving that

$$\text{if} \quad -6 < x < -6+\delta \qquad \text{then} \qquad \left|\sqrt[8]{6+x} - 0\right| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that satisfies $-6 < x < -6+\delta$, or $0 < x+6 < \delta$.

$$\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$$

$$\left|\sqrt[8]{6+x}\right| < \varepsilon$$

$$\sqrt[8]{6+x} < \varepsilon$$

$$\left(\sqrt[8]{6+x}\right)^8 < (\varepsilon)^8$$

$$6+x < \varepsilon^8$$

Choose $\delta = \varepsilon^8$. Now, assuming that $6+x < \delta$,

$$\begin{aligned} \left|\sqrt[8]{6+x} - 0\right| &= \left|\sqrt[8]{6+x}\right| \\ &= \sqrt[8]{6+x} \\ &< \sqrt[8]{\delta} \\ &= \sqrt[8]{\varepsilon^8} \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to -6^+} \sqrt[8]{6+x} = 0.$$